Recovery of small electromagnetic inhomogeneities from boundary measurements in time-dependent Maxwell's equations

Christian Daveau *Abdessatar Khelifi †

February 1, 2008

Abstract

We consider for the time-dependent Maxwell's equations the inverse problem of identifying locations and certain properties of small electromagnetic inhomogeneities in a homogeneous background medium from dynamic measurements of the tangential component of the magnetic field on the boundary (or a part of the boundary) of a domain.

Key words. Maxwell's equations, inhomogeneities, inverse problem, reconstruction

2000 AMS subject classifications. 35R30, 35B40, 35B37, 78M35

1 Introduction

Let Ω be a bounded C^2-domain in $\mathbb{R}^d$, $d = 2, 3$. Assume that Ω contains a finite number of inhomogeneities, each of the form $z_j + \alpha B_j$, where $B_j \subset \mathbb{R}^d$ is a bounded, smooth domain containing the origin. The total collection of inhomogeneities is $\mathcal{B}_\alpha = \cup_{j=1}^m (z_j + \alpha B_j)$. The points $z_j \in \Omega, j = 1, \ldots, m$, which determine the location of the inhomogeneities, are assumed to satisfy the following inequalities:

$$|z_j - z_{j'}| \geq c_0 > 0, \forall\, j \neq j' \quad \text{and} \quad \mathrm{dist}(z_j, \partial\Omega) \geq c_0 > 0, \forall\, j. \tag{1}$$

Assume that $\alpha > 0$, the common order of magnitude of the diameters of the inhomogeneities, is sufficiently small, that these inhomogeneities are disjoint, and that their distance to $\mathbb{R}^d \setminus \overline{\Omega}$ is larger than $c_0/2$. Let μ_0 and ε_0 denote the permeability and the permittivity of the background medium, and assume that $\mu_0 > 0$ and $\varepsilon_0 > 0$ are positive constants. Let $\mu_j > 0$ and $\varepsilon_j > 0$ denote the permeability and the permittivity

*Département de Mathématiques, Site Saint-Martin II, BP 222, & Université de Cergy-Pontoise, 95302 Cergy-Pontoise Cedex, France (Email: christian.daveau@math.u-cergy.fr).

†Département de Mathématiques & Informatique Faculté des Sciences, 7021 Zarzouna - Bizerte, Tunisia (Email:abdessatar.khelifi@fsb.rnu.tn).

of the j-th inhomogeneity, $z_j + \alpha B_j$, these are also assumed to be positive constants. Introduce the piecewise-constant electric permittivity

$$\varepsilon_\alpha(x) = \begin{cases} \varepsilon_0, & x \in \Omega \setminus \bar{\mathcal{B}}_\alpha, \\ \varepsilon_j, & x \in z_j + \alpha B_j, \ j = 1 \dots m. \end{cases} \tag{2}$$

If we allow the degenerate case $\alpha = 0$, then the function $\varepsilon_0(x)$ equals the constant ε_0. Assume that, the magnetic permeability is given by

$$\mu_\alpha(x) = \mu_0, \qquad \text{for all } x \in \Omega. \tag{3}$$

Let $\nu = \nu(x)$ denote the outward unit normal vector to Ω at a point on $\partial\Omega$, and $\partial_t = \frac{\partial}{\partial t}$.

In this paper, we will denote by bold letters the functional spaces for the vector fields. Thus $H^s(\Omega)$ denotes the usual Sobolev space on Ω and $\mathbf{H}^s(\Omega)$ denotes $(H^s(\Omega))^d$ and $\mathbf{L}^2(\Omega)$ denotes $(L^2(\Omega))^d$. As usual for Maxwell equations, we need spaces of fields with square integrable curls:

$$\mathbf{H}(\text{ curl };\Omega) = \{u \in \mathbf{L}^2(\Omega), \text{ curl } u \in \mathbf{L}^2(\Omega)\},$$

and with square integrable divergences

$$\mathbf{H}(\text{div };\Omega) = \{u \in \mathbf{L}^2(\Omega), \text{div } u \in L^2(\Omega)\}.$$

We will also need the following functional spaces:

$$Y(\Omega) = \{u \in \mathbf{L}^2(\Omega), \text{div } u = 0 \text{ in } \Omega\}, \quad X(\Omega) = \mathbf{H}^1(\Omega) \cap Y(\Omega),$$

and $TL^2(\partial\Omega)$ the space of vector fields on $\partial\Omega$ that lie in $\mathbf{L}^2(\partial\Omega)$. Finally, the "minimal" choice for the magnetic variational space would be

$$X_N(\Omega) = \{v \in \mathbf{H}(\text{ curl };\Omega) \cap \mathbf{H}(\text{div };\Omega); \quad v \times \nu = 0 \quad \text{on } \partial\Omega\}.$$

We use $\langle \cdot, \cdot \rangle$ for the duality bracket and $(\cdot, \cdot)$ for the $\mathbf{L}^2$ product.

In the non flawed region Ω an electric field/magnetic field pair (E, H) satisfies:

$$\begin{cases} \text{curl } E = -\mu_0 \partial_t H & \text{in } \Omega \times (0,T), \\ \text{curl } H = \varepsilon_0 \partial_t E & \text{in } \Omega \times (0,T), \end{cases} \tag{4}$$

Let $TH^{-1/2}_{\text{div}}(\partial\Omega)$ denote the space of tangential vector fields on $\partial\Omega$ that lie in $H^{-1/2}(\partial\Omega)$. The most common boundary data is

$$(E \times \nu)|_{\partial\Omega \times (0,T)} \qquad \text{given in } TH^{-1/2}_{\text{div}}(\partial\Omega). \tag{5}$$

By dividing the second equation in (4) by ε_0 and taking the curl, we obtain in terms of the magnetic field:

$$\text{curl } (\frac{1}{\varepsilon_0} \text{ curl } H) + \mu_0 \partial_t^2 H = 0 \quad \text{in } \Omega \times (0,T), \tag{6}$$

and the boundary data is supposed to be given by

$$(H \times \nu)|_{\partial\Omega \times (0,T)} = f \qquad \text{given in } TH^{-1/2}_{\text{div}}(\partial\Omega). \tag{7}$$

Moreover, we set

$$H|_{t=0} = \varphi, \partial_t H|_{t=0} = \psi \quad \text{in } \Omega. \tag{8}$$

Here $T > 0$ is a final observation time and $\varphi, \psi \in \mathcal{C}^\infty(\overline{\Omega})$ and $f \in \mathcal{C}^\infty(0, T; \mathcal{C}^\infty(\partial\Omega))$ are subject to the compatibility conditions

$$\partial_t^{2l} f|_{t=0} = (\Delta^l \varphi) \times \nu|_{\partial\Omega} \text{ and } \partial_t^{2l+1} f|_{t=0} = (\Delta^l \psi) \times \nu|_{\partial\Omega}, \quad l = 1, 2, \ldots$$

Let $H_\alpha \in \mathbb{R}^d$ be the magnetic field corresponding to the case of the presence of a finite number of small electromagnetic inhomogeneities. This field (under the assumption (3)) satisfies

$$\begin{cases} \text{curl } (\frac{1}{\varepsilon_\alpha} \text{curl } H_\alpha) + \mu_0 \partial_t^2 H_\alpha = 0 \quad \text{in } \Omega \times (0, T), \\ \text{div } (\mu_0 H_\alpha) = 0 \quad \text{in } \Omega \times (0, T), \\ H_\alpha|_{t=0} = \varphi, \partial_t H_\alpha|_{t=0} = \psi \quad \text{in } \Omega, \\ H_\alpha \times \nu|_{\partial\Omega \times (0,T)} = f, \end{cases} \tag{9}$$

It is well known that (6) has a unique solution $H \in \mathcal{C}^\infty([0, T] \times \overline{\Omega})$. It is also known (see for example [15]) that since Ω is smooth ($\mathcal{C}^2-$ regularity would be sufficient) the non homogeneous Maxwell's equations (9) have a unique weak solution $H_\alpha \in \mathcal{C}^0(0, T; X(\Omega)) \cap \mathcal{C}^1(0, T; \mathbf{L}^2(\Omega))$. Indeed, curl H_α belongs to $\mathcal{C}^0(0, T; X(\Omega)) \cap \mathcal{C}^1(0, T; \mathbf{L}^2(\Omega))$.

Having found H_α, we then obtain the field E_α through the formula:

$$\partial_t E_\alpha = \frac{1}{\varepsilon_\alpha} \text{ curl } H_\alpha.$$

Our main goal in this paper is to determine, most effectively, properties of the inhomogeneities $z_j + \alpha B_j$, from over determined boundary information about specific solutions to (9). In particular, we study media that consist of a homogeneous(constant coefficient) electromagnetic material with a finite number of small inhomogeneities, and as our main result we derive asymptotic formulas for the perturbations in the (tangential) boundary magnetic fields caused by the presence of these inhomogeneities. Our formulas may be used to determine properties (location, relative size) of the small inhomogeneities in case a single, or a few (tangential) boundary electric fields and their corresponding(tangential)boundary magnetic fields are known. For stationary Maxwell's equations it has been known that the Dirichlet to Neumann map uniquely determines (smooth) isotropic electromagnetic parameters, see [14], [16], [18]. We will provide in this paper a rigorous derivation of the inverse Fourier transform of a linear combination of derivatives of point masses, located at the positions z_j of the inhomogeneities, as the leading order term of an appropriate averaging of (partial) dynamic boundary measurements of the tangential components of magnetic fields on part of the boundary. We refer the reader to [17],[19], [6], and [8] for discussions on closely related (stationary) identification problems.

Our approach, aimed at determining specific internal features of an object based on electromagnetic boundary measurements, differs from [1], [2], [3], [4], [20] and it can be regarded as constructive method, but until tested.

2 Asymptotic behavior

We start the derivation of the asymptotic formula for $\operatorname{curl} H_\alpha \times \nu$ with the following estimate.

Lemma 2.1 *The following estimate as $\alpha \to 0$ holds:*

$$||\partial_t(H_\alpha - H)||_{L^\infty(0,T;\mathbf{L}^2(\Omega))} + ||H_\alpha - H||_{L^\infty(0,T;X_N(\Omega))} \le C\alpha^d, \tag{10}$$

where the constant C is independent of α and the set of points $\{z_j\}_{j=1}^m$ provided that assumption (1) holds.

Proof. From (6)-(9), it is obvious that $H_\alpha - H \in X_N(\Omega)$, then due to the Green formula we have for any $\mathbf{v} \in X_N(\Omega)$:

$$\int_\Omega \mu_0 \partial_t^2 (H_\alpha - H)\cdot \mathbf{v} + \int_\Omega \frac{1}{\varepsilon_\alpha} \operatorname{curl}(H_\alpha - H)\cdot \operatorname{curl}\mathbf{v} = \sum_{j=1}^m (\frac{1}{\varepsilon_0} - \frac{1}{\varepsilon_j}) \int_{z_j+\alpha B_j} \operatorname{curl} H \cdot \operatorname{curl}\mathbf{v}. \tag{11}$$

Let $\mathbf{v}_\alpha$ be defined by

$$\begin{cases} \mathbf{v}_\alpha \in X_N(\Omega), \\ \operatorname{curl} \frac{1}{\varepsilon_\alpha} \operatorname{curl} \mathbf{v}_\alpha = \partial_t (H_\alpha - H) \quad \text{in } \Omega. \end{cases} \tag{12}$$

Then,

$$\int_\Omega \frac{1}{\varepsilon_\alpha} \operatorname{curl}(H_\alpha - H)\cdot \operatorname{curl}\mathbf{v}_\alpha = -\int_\Omega \partial_t (H_\alpha - H)\cdot (H_\alpha - H) = -\frac{1}{2}\partial_t \int_\Omega |H_\alpha - H|^2$$

and by Green formula, relation (12) gives:

$$\begin{aligned} \int_\Omega \partial_t^2 (H_\alpha - H)\cdot \mathbf{v}_\alpha &= \int_\Omega \operatorname{curl} \frac{1}{\varepsilon_\alpha} \operatorname{curl} \partial_t \mathbf{v}_\alpha \cdot \mathbf{v}_\alpha \\ &= -\int_\Omega \frac{1}{\varepsilon_\alpha} \operatorname{curl} \partial_t \mathbf{v}_\alpha \cdot \operatorname{curl}\mathbf{v}_\alpha \\ &= -\frac{1}{2}\partial_t \int_\Omega \frac{1}{\varepsilon_\alpha} |\operatorname{curl}\mathbf{v}_\alpha|^2. \end{aligned}$$

Thus, it follows from (11) that

$$\mu_0 \partial_t \int_\Omega \frac{1}{\varepsilon_\alpha} |\operatorname{curl}\mathbf{v}_\alpha|^2 + \partial_t \int_\Omega |H_\alpha - H|^2 = -2\sum_{j=1}^m (\frac{1}{\varepsilon_0} - \frac{1}{\varepsilon_j}) \int_{z_j+\alpha B_j} \operatorname{curl} H \cdot \operatorname{curl}\mathbf{v}_\alpha.$$

Next,

$$|\sum_{j=1}^m (\frac{1}{\varepsilon_0} - \frac{1}{\varepsilon_j}) \int_{z_j+\alpha B_j} \operatorname{curl} H \cdot \operatorname{curl}\mathbf{v}_\alpha| \le C || \operatorname{curl} H||_{\mathbf{L}^2(\mathcal{B}_\alpha)} || \operatorname{curl}\mathbf{v}_\alpha||_{\mathbf{L}^2(\Omega)}.$$

Since $H \in \mathcal{C}^\infty([0,T]\times\overline{\Omega})$ we have

$$|| \operatorname{curl} H||_{\mathbf{L}^2(\mathcal{B}_\alpha)} \le || \operatorname{curl} H||_{L^\infty(\mathcal{B}_\alpha)} \alpha^d (\sum_{j=1}^m |B_j|)^{\frac{1}{2}} \le C\alpha^d,$$

which gives

$$|\sum_{j=1}^{m}(\frac{1}{\varepsilon_0}-\frac{1}{\varepsilon_j})\int_{z_j+\alpha B_j} \text{ curl } H\cdot \text{ curl } \mathbf{v}_\alpha| \le C\alpha^d || \text{ curl } \mathbf{v}_\alpha||_{\mathbf{L}^2(\Omega)}$$

and so,

$$\mu_0\partial_t\int_\Omega\frac{1}{\varepsilon_\alpha}| \text{ curl } \mathbf{v}_\alpha|^2+\partial_t\int_\Omega|H_\alpha-H|^2 \le C\alpha^d(\int_\Omega\frac{1}{\varepsilon_\alpha}| \text{ curl } \mathbf{v}_\alpha|^2+\int_\Omega|H_\alpha-H|^2)^{1/2}. \quad (13)$$

From the Gronwall Lemma it follows that

$$(\int_\Omega\frac{1}{\varepsilon_\alpha}| \text{ curl } \mathbf{v}_\alpha|^2)^{1/2}+(\int_\Omega|H_\alpha-H|^2))^{1/2}\le C\alpha^d. \quad (14)$$

Combining (14) with the fact that

$$||\partial_t(H_\alpha-H)||_{L^\infty(0,T;H^{-1}(\Omega))}\le C|| \text{ curl } \mathbf{v}_\alpha||_{L^\infty(0,T;\mathbf{L}^2(\Omega))},$$

the following estimate holds

$$||H_\alpha-H||_{L^\infty(0,T;\mathbf{L}^2(\Omega))}+||\partial_t(H_\alpha-H)||_{L^\infty(0,T;\mathbf{L}^2(\Omega))}\le C\alpha^d. \quad (15)$$

Now, taking (formally) $\mathbf{v}=\partial_t(H_\alpha-H)$ in (11) we arrive at

$$\mu_0\partial_t\int_\Omega\Big[|\partial_t(H_\alpha-H)|^2+\frac{1}{\varepsilon_\alpha}| \text{ curl } (H_\alpha-H)|^2\Big]=2\sum_{j=1}^{m}(\frac{1}{\varepsilon_0}-\frac{1}{\varepsilon_j})\int_{z_j+\alpha B_j} \text{ curl } H\cdot \text{ curl } \partial_t(H_\alpha-H).$$

By using the regularity of H in Ω and estimate (15) given above, we see that

$$|\sum_{j=1}^{m}(\frac{1}{\varepsilon_0}-\frac{1}{\varepsilon_j})\int_{z_j+\alpha B_j} \text{ curl } H\cdot \text{ curl } \partial_t(H_\alpha-H)|\le C|| \text{ curl } H||_{\mathbf{H}^2(\mathcal{B}_\alpha)}||\partial_t(H_\alpha-H)||_{\mathbf{H}^{-1}(\Omega)}\le C\alpha^{2d},$$

where C is independent of t and α, and so, we obtain

$$\partial_t\int_\Omega\Big[|\partial_t(H_\alpha-H)|^2+\frac{1}{\varepsilon_\alpha}| \text{ curl } (H_\alpha-H)|^2\Big]\le C\alpha^{2d}$$

which yields the following estimate

$$||\partial_t(H_\alpha-H)||_{L^\infty(0,T;\mathbf{L}^2(\Omega))}+||H_\alpha-H||_{L^\infty(0,T;X_N(\Omega))}\le C\alpha^d,$$

where C is independent of α and the points $\{z_j\}_{j=1}^m$.

□

Now, we can estimate curl $H_\alpha -$ curl H as follows.

Proposition 2.1 *Let H_α and H be solutions to the problems (9) and (6) respectively. There exist constants $0<\alpha_0$, C such that for $0<\alpha<\alpha_0$ the following estimate holds:*

$$|| \text{ curl } (H_\alpha-H)||_{L^\infty(0,T;\mathbf{L}^2(\Omega))}\le C\alpha^d, \quad (16)$$

Proof. To prove estimate (16) it is useful to introduce the following function

$$\hat{v}(x) = \int_0^T v(x,t)z(t)\,dt \in L^2(\Omega), \tag{17}$$

where $v \in L^1(0,T;L^2(\Omega))$ and $z(t)$ is a given function in $\mathcal{C}_0^\infty(]0,T[)$.
Then,

$$\hat{H}(x) = \int_0^T H(x,t)z(t)\,dt \text{ and } \hat{H}_\alpha(x) = \int_0^T H_\alpha(x,t)z(t)\,dt \in X(\Omega),$$

which by relation (10) give

$$\begin{cases} (\hat{H}_\alpha - \hat{H}) \in \mathbf{H}^1(\Omega), \\ \text{curl curl } (\hat{H}_\alpha - \hat{H}) = 0(\alpha^d) \quad \text{in } \Omega, \\ \text{div } (\hat{H}_\alpha - \hat{H}) = 0 \quad \text{in } \Omega, \\ (\hat{H}_\alpha - \hat{H}) \times \nu|_{\partial\Omega} = 0, \end{cases}$$

and so,

$$||\text{ curl } (\hat{H}_\alpha - \hat{H})||_{\mathbf{L}^2(\Omega)} = O(\alpha^d). \tag{18}$$

The fact that curl $(H_\alpha - H)$ belongs to $L^\infty(0,T;\mathbf{L}^2(\Omega))$ and by using (17) and (18) we arrive at:

$$\int_\Omega |\text{ curl } H_\alpha(x,t) - \text{ curl } H(x,t)|^2\,dx = O(\alpha^{2d}) \quad \text{ a.e. in } t \in (0,T),$$

which means that

$$||\text{ curl } (H_\alpha - H)||_{\mathbf{L}^2(\Omega)} = O(\alpha^d) \quad \text{ a.e. in } t \in (0,T).$$

This equation can be bounded easily according to $t \in (0,T)$. Thus, estimate (16) holds.
□

Before formulating our main result in this section, let us denote $\Phi_j, j = 1, \ldots, m$ the unique vector-valued solution of the following free space Laplace equation:

$$\begin{cases} \Delta\Phi_j = 0 \text{ in } B_j, \text{ and } \mathbb{R}^d \setminus \overline{B_j}, \\ \Phi_j \text{ is continuous across } \partial B_j, \\ \dfrac{\varepsilon_j}{\varepsilon_0}\dfrac{\partial\Phi_j}{\partial\nu_j}|_+ - \dfrac{\partial\Phi_j}{\partial\nu_j}|_- = -\nu_j, \\ \lim\limits_{|y|\to+\infty} |\Phi_j(y)| = 0, \end{cases} \tag{19}$$

where ν_j denotes the outward unit normal to ∂B_j, and superscripts $-$ and $+$ indicate the limiting values as the point approaches ∂B_j from outside B_j, and from inside B_j, respectively. The existence and uniqueness of this Φ_j can be established using single layer potentials with suitably chosen densities, see [6] for the case of conductivity problem. For each inhomogeneity $z_j + \alpha B_j$ we introduce the polarizability tensor

M_j which is a $d \times d$, symmetric, positive definite matrix associated with the j-th inhomogeneity, given by

$$(M_j)_{k,l} = e_k \cdot (\int_{\partial B_j} (\nu_j + (\frac{\varepsilon_j}{\varepsilon_0} - 1)\frac{\partial \Phi_j}{\partial \nu_j}|_{+}(y))y \cdot e_l \, d\sigma_j(y)). \tag{20}$$

Here $(e_1, \ldots, e_d)$ is an orthonormal basis of $\mathbb{R}^d$. In terms of this function we are able to prove the following result about the asymptotic behavior of $\text{ curl } H_\alpha \cdot \nu_j|_{\partial(z_j+\alpha B_j)^+}$.

Theorem 2.1 *Suppose that (1) is satisfied and let $\Phi_j, j = 1, \ldots, m$ be given as in (19). Then, for the solutions H_α, H of problems (9) and (6) respectively, and for $y \in \partial B_j$ we have*

$$\begin{aligned} (\text{ curl } H_\alpha(z_j + \alpha y) \cdot \nu_j)|_{\partial(z_j+\alpha B_j)^+} = & \text{ curl } H(z_j, t) \cdot \nu_j \\ & + (1 - \frac{\varepsilon_j}{\varepsilon_0})\frac{\partial \Phi_j}{\partial \nu_j}|_{+}(y) \cdot \text{ curl } H(z_j, t) + o(1). \end{aligned} \tag{21}$$

The term $o(1)$ uniform in $y \in \partial B_j$ and $t \in (0, T)$ and depends on the shape of $\{B_j\}_{j=1}^m$ and Ω, the constants c_0, T, ε_0, $\{\varepsilon_j\}_{j=1}^m$, the data φ, ψ, and f, but is otherwise independent of the points $\{z_j\}_{j=1}^m$.

Proof.
Let $\mathcal{H}_\alpha = \text{ curl } H_\alpha(x, t)$ and $\mathcal{H}_0 = \text{ curl } H(x, t)$. Then, according to (6)-(9) we have

$$\mu_0 \partial_t^2 H_\alpha - \text{ curl } \frac{1}{\varepsilon_\alpha}\mathcal{H}_\alpha = 0 \text{ and } \text{ curl } \mathcal{H}_\alpha = 0, \text{ for } x \in \Omega. \tag{22}$$

We restrict, for simplicity, our attention to the case of a single inhomogeneity, i.e., the case $m = 1$. The proof for any fixed number m of well separated inhomogeneities follows by iteration of the argument that we will present for the case $m = 1$. In order to further simplify notation, we assume that the single inhomogeneity has the form αB, that is, we assume it is centered at the origin. We denote the electromagnetic permeability inside αB by ε_* and define Φ_* the same as Φ_j, defined in (19), but with B_j and ε_j replaced by B and ε_*, respectively. Define ν to be the outward unit normal to ∂B. Now, following a common practice in multiscale expansions we introduce the local variable $y = \dfrac{x}{\alpha}$, then the domain $\tilde{\Omega} = (\dfrac{\Omega}{\alpha})$ is well defined.
Next, let ϖ be given in $\mathcal{C}_0^\infty(]0, T[)$. For any function $v \in \mathbf{L}^1(0, T; \mathbf{L}^2(\Omega))$, we define

$$\hat{v}(x) = \int_0^T v(x, t) \, \varpi(t) \, dt \in \mathbf{L}^2(\Omega).$$

We remark that $\widehat{\partial_t v}(x) = -\displaystyle\int_0^T v(x, t)\varpi'(t) \, dt$. So that we deduce from (22) that $\hat{\mathcal{H}}_\alpha$ satisfies

$$\begin{cases} \text{curl } \dfrac{1}{\varepsilon_\alpha}\hat{\mathcal{H}}_\alpha = \displaystyle\int_0^T H_\alpha \, \varpi''(t) \, dt & \text{in } \Omega, \\ \text{curl } \hat{\mathcal{H}}_\alpha = 0 & \text{in } \Omega. \end{cases}$$

Analogously, $\hat{\mathcal{H}}$ satisfies

$$\begin{cases} \dfrac{1}{\varepsilon_0} \text{ curl } \hat{\mathcal{H}} = \displaystyle\int_0^T H\, \varpi''(t)\, dt \quad \text{in } \Omega, \\ \text{curl } \hat{\mathcal{H}} = 0 \quad \text{in } \Omega. \end{cases}$$

Indeed, we have $\hat{\mathcal{H}}_\alpha \times \nu = \hat{\mathcal{H}} \times \nu = \text{ curl }_{\partial\Omega} \hat{f} \times \nu$ on the boundary $\partial\Omega$, where $\text{ curl }_{\partial\Omega}$ is the tangential curl. Following [4] and [1], we introduce q^*_α as the unique solution to the following problem

$$\begin{cases} \Delta q^*_\alpha = 0 \quad \text{in } \tilde{\Omega} = (\dfrac{\Omega}{\alpha}) \setminus \overline{B} \text{ and in } B, \\ q^*_\alpha \text{ is continuous across } \partial B, \\ \varepsilon_0 \dfrac{\partial q^*_\alpha}{\partial \nu}|_+ - \varepsilon_* \dfrac{\partial q^*_\alpha}{\partial \nu}|_- = -(\varepsilon_0 - \varepsilon_*) \hat{\mathcal{H}}(\alpha y) \cdot \nu \quad \text{on } \partial B, \\ q^*_\alpha = 0 \quad \text{on } \partial\tilde{\Omega}. \end{cases}$$

The jump condition

$$\varepsilon_0 \frac{\partial q^*_\alpha}{\partial \nu}|_+ - \varepsilon_* \frac{\partial q^*_\alpha}{\partial \nu}|_- = -(\varepsilon_0 - \varepsilon_*) \hat{\mathcal{H}}(\alpha y) \cdot \nu \quad \text{on } \partial B$$

guarantees that $\hat{\mathcal{H}}_\alpha(x) - \hat{\mathcal{H}}(x) - \text{ grad }_y q^*_\alpha(\frac{x}{\alpha})$ belongs to the functional space $X_N(\Omega)$, where $\text{ grad }_{\partial\Omega}$ is the tangential gradient. Since

$$\begin{cases} \text{curl } \dfrac{1}{\varepsilon_\alpha} (\hat{\mathcal{H}}_\alpha - \hat{\mathcal{H}} - \text{ grad }_y q^*_\alpha(\dfrac{x}{\alpha})) = \displaystyle\int_0^T \Big[H_\alpha - \chi(\Omega \setminus \overline{\alpha B}) H + \frac{\varepsilon_*}{\varepsilon_0} \chi(\alpha B) H \Big] \varpi''(t)\, dt \quad \text{in } \Omega, \\ \text{curl } (\hat{\mathcal{H}}_\alpha - \hat{\mathcal{H}} - \text{ grad }_y q^*_\alpha(\dfrac{x}{\alpha})) = 0 \quad \text{in } \Omega, \\ (\hat{\mathcal{H}}_\alpha - \hat{\mathcal{H}} - \text{ grad }_y q^*_\alpha(\frac{x}{\alpha})) \times \nu = 0 \quad \text{on } \partial\Omega, \end{cases}$$

where $\chi(\omega)$ is the characteristic function of the domain ω, we arrive, as a consequence of the energy estimate given by Lemma 2.1, at the following

$$\begin{cases} (\hat{\mathcal{H}}_\alpha - \hat{\mathcal{H}} - \text{ grad }_y q^*_\alpha(\frac{x}{\alpha})) \in X_N(\Omega), \\ \text{curl } \dfrac{1}{\varepsilon_\alpha} (\hat{\mathcal{H}}_\alpha - \hat{\mathcal{H}} - \text{ curl }_y q^*_\alpha(\dfrac{x}{\alpha})) = 0(\alpha) \quad \text{in } \Omega, \\ \text{curl } (\hat{\mathcal{H}}_\alpha - \hat{\mathcal{H}} - \text{ grad }_y q^*_\alpha(\dfrac{x}{\alpha})) = 0 \quad \text{in } \Omega, \\ (\hat{\mathcal{H}}_\alpha - \hat{\mathcal{H}} - \text{ grad }_y q^*_\alpha(\frac{x}{\alpha})) \times \nu = 0 \quad \text{on } \partial\Omega. \end{cases}$$

From [4] we know that this yields the following estimate

$$|| \text{ curl } \frac{1}{\varepsilon_\alpha} (\hat{\mathcal{H}}_\alpha - \hat{\mathcal{H}} - \text{ grad }_y q^*_\alpha(\frac{x}{\alpha}))||_{L^2(\Omega)} + ||\hat{\mathcal{H}}_\alpha - \hat{\mathcal{H}} - \text{ grad }_y q^*_\alpha(\frac{x}{\alpha})||_{L^2(\Omega)} \leq C\alpha,$$

and so,

$$(\hat{\mathcal{H}}_\alpha - \hat{\mathcal{H}} - \text{ grad }_y q^*_\alpha(\frac{x}{\alpha})) \cdot \nu|_+ = 0(\alpha) \quad \text{on } \partial(\alpha B).$$

Now, we denote by q_* be the unique (scalar) solution to

$$\begin{cases} \Delta q_* = 0 \quad \text{in } \mathbb{R}^d \setminus \overline{B} \text{ and in } B, \\ q_* \text{ is continuous across } \partial B, \\ \varepsilon_0 \dfrac{\partial q_*}{\partial \nu}|_+ - \varepsilon_* \dfrac{\partial q_*}{\partial \nu}|_- = -(\varepsilon_0 - \varepsilon_*)\hat{\mathcal{H}}(0) \cdot \nu \quad \text{on } \partial B, \\ \lim\limits_{|y| \to +\infty} q_* = 0. \end{cases}$$

In the spirit of Theorem 1 in [6] it follows that

$$||(\text{ grad }_y q_* - \text{ grad }_y q_\alpha^*)(\frac{x}{\alpha})||_{\mathbf{L}^2(\Omega)} \leq C\alpha^{1/2},$$

which yields

$$(\hat{\mathcal{H}}_\alpha - \hat{\mathcal{H}} - \text{ grad }_y q_*(\frac{x}{\alpha})) \cdot \nu = o(1) \quad \text{on } \partial(\alpha B).$$

Writing q_* in terms of Φ_* gives

$$\int_0^T \Big[(\text{ curl } H_\alpha(\alpha y)\cdot\nu)|_{\partial(\alpha B)^+} - \nu\cdot \text{ curl } H(0,t) - (\frac{\varepsilon_0}{\varepsilon_*} - 1)\frac{\partial \Phi_*}{\partial \nu}|_+(y)\cdot \text{ curl } H(0,t)\Big]\varpi(t)\, dt = o(1),$$

for any $\varpi \in \mathcal{C}_0^\infty(]0,T[)$, and so, by iterating the same argument for the case of m (well separated) inhomogeneities $z_j + \alpha B_j, j = 1, \dots, m$, we arrive at the promised asymptotic formula (21).

□

3 Reconstruction

Before describing our identification and reconstruction procedure, let us introduce the following cutoff function $\beta(x) \in \mathcal{C}_0^\infty(\Omega)$ such that $\beta \equiv 1$ in a subdomain Ω' of Ω that contains the inhomogeneities $\mathcal{B}_\alpha$ and let $\eta \in \mathbb{R}^d$. We will take in what follows $H(x,t) = \eta^\perp e^{i\eta\cdot x - i\sqrt{\varepsilon_0}|\eta|t}$ where $\eta^\perp$ is a unit vector that is orthogonal to η which corresponds to taking $\varphi(x) = \eta^\perp e^{i\eta\cdot x}, \psi(x) = -i\sqrt{\varepsilon_0}|\eta|\eta^\perp e^{i\eta\cdot x}$, and $f(x,t) = \eta^\perp \times \nu e^{i\eta\cdot x - i\sqrt{\varepsilon_0}|\eta|t}$ and assume that we are in possession of the measurements of:

$$\text{curl } H_\alpha \times \nu \quad \text{ on } \Gamma \times (0,T),$$

where Γ is an open part of $\partial\Omega$. Suppose now that T and the part Γ of the boundary $\partial\Omega$ are such that they geometrically control Ω which means that they satisfy the geometric control hypothesis of the work of Bardos, Lebeau and Rauch in [5]:

Definition 3.1 *Let Γ be an open subset of $\partial\Omega$ and T a positive number. One says that (Γ, T) geometrically control Ω if for every geometrical optic ray $s \mapsto \gamma(s)$, there exists $s_0 \in]0,T[$ such that $\gamma(s_0) \in \Gamma \times]0,T[$ and $\gamma(s_0)$ non diffractive point.*

It follows from [15] (see also [11], [9] and [10]) that we can construct (a unique) $g_\eta \in H_0^1(0,T; TL^2(\Gamma))$ (by the Hilbert Uniqueness Method) such that the unique weak

solution w_η to

$$\begin{cases} (\partial_t^2 + \text{ curl } \text{ curl })w_\eta = 0 \quad \text{in } \Omega \times (0,T), \\ \text{div } w_\eta = 0 \quad \text{in } \Omega \times (0,T), \\ w_\eta|_{t=0} = \beta(x)\eta^\perp e^{i\eta\cdot x}, \partial_t w_\eta|_{t=0} = 0 \quad \text{in } \Omega, \\ w_\eta \times \nu|_{\partial\Omega\setminus\overline{\Gamma}\times(0,T)} = 0, \\ w_\eta \times \nu|_{\Gamma\times(0,T)} = g_\eta, \end{cases} \tag{23}$$

satisfies $w_\eta(T) = \partial_t w_\eta(T) = 0$ in Ω.

Let $\theta_\eta \in H^1(0,T;TL^2(\Gamma))$ denote the unique solution of the Volterra equation of second kind

$$\begin{cases} \partial_t\theta_\eta(x,t) + \displaystyle\int_t^T e^{-i|\eta|(s-t)}(\theta_\eta(x,s) - i|\eta|\partial_t\theta_\eta(x,s))\, ds = g_\eta(x,t) \quad \text{for } x \in \Gamma, t \in (0,T), \\ \theta_\eta(x,0) = 0 \quad \text{for } x \in \Gamma. \end{cases} \tag{24}$$

The existence and uniqueness of this θ_η in $\mathbf{H}^1(0,T;TL^2(\Gamma))$ for any $\eta \in \mathbb{R}^d$ can be established using the resolvent kernel. However, observing from differentiation of (24) with respect to t that θ_η is the unique solution of the ODE:

$$\begin{cases} \partial_t^2\theta_\eta - \theta_\eta = e^{i|\eta|t}\partial_t(e^{-i|\eta|t}g_\eta) \quad \text{for } x \in \Gamma, t \in (0,T), \\ \theta_\eta(x,0) = 0, \partial_t\theta_\eta(x,T) = 0 \quad \text{for } x \in \Gamma, \end{cases} \tag{25}$$

the function θ_η may be find (in practice) explicitly with variation of parameters and it also immediately follows from this observation that θ_η belongs to $\mathbf{H}^2(0,T;TL^2(\Gamma))$. We introduce v_η as the unique weak solution (obtained by transposition as done in [13] and in [12] [Theorem 4.2, page 46] for the scalar function) in $\mathcal{C}^0(0,T;X(\Omega)) \cap \mathcal{C}^1(0,T;L^2(\Omega))$ to the following problem

$$\begin{cases} (\partial_t^2 + \text{ curl } \text{ curl })v_\eta = 0 \quad \text{in } \Omega \times (0,T), \\ \text{div } v_\eta = 0 \quad \text{in } \Omega \times (0,T), \\ v_\eta|_{t=0} = 0 \quad \text{in } \Omega, \\ \partial_t v_\eta|_{t=0} = \displaystyle\sum_{j=1}^m i(1 - \frac{\varepsilon_0}{\varepsilon_j})\eta \times (\nu_j + (\frac{\varepsilon_0}{\varepsilon_j} - 1)\frac{\partial\Phi_j}{\partial\nu_j}|_+)e^{i\eta\cdot z_j}\delta_{\partial(z_j+\alpha B_j)} \in Y(\Omega) \quad \text{in } \Omega, \\ v_\eta \times \nu|_{\partial\Omega\times(0,T)} = 0. \end{cases}$$

Then, the following holds.

Proposition 3.1 *Suppose that Γ and T geometrically control Ω. For any $\eta \in \mathbb{R}^d$ and $\eta^\perp$ unit vector in $\mathbb{R}^d$ that is orthogonal to η, we have*

$$\int_0^T \int_\Gamma g_\eta \cdot (\text{ curl } v_\eta \times \nu) = \alpha^d \sum_{j=1}^m \varepsilon_0(1 - \frac{\varepsilon_j}{\varepsilon_0})e^{2i\eta\cdot z_j}\Big(\eta \times (\int_{\partial B_j}(\nu_j \tag{26}$$

$$+(\frac{\varepsilon_j}{\varepsilon_0} - 1)\frac{\partial\Phi_j}{\partial\nu_j}|_+(y))))y\cdot\eta\Big)\cdot\eta^\perp\, ds_j(y) + O(\alpha^d).$$

Proof. Multiply the equation $(\partial_t^2 + \text{ curl } \text{ curl })v_\eta = 0$ by w_η and integrating by parts in $t \in (0,T)$, we get

$$\int_0^T \int_\Omega (\partial_t^2 + \text{ curl } \text{ curl })v_\eta w_\eta = \int_0^T \int_\Omega \text{ curl } \text{ curl } v_\eta w_\eta + \int_\Omega \int_0^T \partial_t^2 v_\eta w_\eta$$

$$= \int_0^T \int_\Omega \text{ curl } \text{ curl } v_\eta w_\eta + \int_\Omega \partial_t v_\eta w_\eta|_{t=0} - \partial_t v_\eta w_\eta|_{t=T} - \int_\Omega \int_0^T \partial_t v_\eta \partial_t w_\eta$$

$$= \int_0^T \int_\Omega \text{ curl } \text{ curl } v_\eta w_\eta + \int_\Omega \partial_t v_\eta w_\eta|_{t=0} + \int_\Omega v_\eta \partial_t w_\eta|_{t=0} + \int_\Omega \int_0^T v_\eta \partial_t^2 w_\eta.$$

So, by Green's formula,

$$\int_0^T \int_\Omega (\partial_t^2 + \text{ curl } \text{ curl })v_\eta w_\eta = -\alpha^{d-1} \sum_{j=1}^m i(1 - \frac{\varepsilon_j}{\varepsilon_0}) e^{2i\eta\cdot z_j} \eta \times (\int_{\partial B_j} (\nu_j +$$

$$(\frac{\varepsilon_j}{\varepsilon_0} - 1)\frac{\partial \Phi_j}{\partial \nu_j}|_+(y)) e^{i\alpha\eta\cdot y}) \cdot \beta(y)\eta^\perp \, ds_j(y)$$

$$-\varepsilon_0^{-1} \int_0^T \int_\Gamma g_\eta \cdot (\text{ curl } v_\eta \times \nu) = 0.$$

Therefore

$$\alpha^{d-1} \sum_{j=1}^m i(1 - \frac{\varepsilon_j}{\varepsilon_0}) e^{2i\eta\cdot z_j} \eta \times (\int_{\partial B_j} (\nu_j + (\frac{\varepsilon_j}{\varepsilon_0} - 1)\frac{\partial \Phi_j}{\partial \nu_j}|_+(y)) e^{i\alpha\eta\cdot y}) \cdot \beta(y)\eta^\perp \, ds_j(y) =$$

$$-\varepsilon_0^{-1} \int_0^T \int_\Gamma g_\eta \cdot (\text{ curl } v_\eta \times \nu).$$

Now, we take the Taylor expansion of $\alpha^{d-1} e^{i\alpha\eta\cdot y}$ in the left side of the last equation and we use the definition of the cutoff function $\beta(x)$, we obtain the convenient asymptotic formula (26).

□

To identify the locations and certain properties of the small inhomogeneities $\mathcal{B}_\alpha$ let us view the averaging of the boundary measurements

$$\text{curl } H_\alpha \times \nu|_{\Gamma\times(0,T)},$$

using the solution θ_η to the Volterra equation (24) or equivalently the ODE (25), as a function of η. The following holds.

Theorem 3.1 *Let $\eta \in \mathbb{R}^d$ and $\eta^\perp$ be a unit vector in $\mathbb{R}^d$ that is orthogonal to η. Let H_α be the unique solution in $\mathcal{C}^0(0,T;X(\Omega)) \cap \mathcal{C}^1(0,T;L^2(\Omega))$ to the Maxwell's equations (9) with $\varphi(x) = \eta^\perp e^{i\eta\cdot x}$, $\psi(x) = -i\sqrt{\varepsilon_0}|\eta|\eta^\perp e^{i\eta\cdot x}$, and $f(x,t) = \eta^\perp e^{i\eta\cdot x - i\sqrt{\varepsilon_0}|\eta|t}$. Suppose that Γ and T geometrically control Ω, then we have*

$$\int_0^T \int_\Gamma \Big[\theta_\eta \cdot (\text{ curl } H_\alpha \times \nu - \text{ curl } H \times \nu) + \partial_t \theta_\eta \cdot \partial_t(\text{ curl } H_\alpha \times \nu - \text{ curl } H \times \nu)\Big] =$$

$$\alpha^d \sum_{j=1}^m (\varepsilon_0 - \varepsilon_j) e^{2i\eta\cdot z_j} (\eta \times M_j(\eta)) \cdot \eta^\perp \ + O(\alpha^d),$$

(27)

where θ_η is the unique solution to the Volterra equation (25) with g_η defined as the boundary control in (23) and M_j is the polarization tensor of B_j, defined by

$$(M_j)_{k,l} = e_k \cdot (\int_{\partial B_j} (\nu_j + (\frac{\varepsilon_j}{\varepsilon_0} - 1)\frac{\partial \Phi_j}{\partial \nu_j}|_+(y)) y \cdot e_l \, ds_j(y)). \tag{28}$$

Here $(e_1, \cdots, e_d)$ is an orthonormal basis of $\mathbb{R}^d$. The term $O(\alpha^d)$ is independent of the points $\{z_j, \quad j = 1, \cdots, m\}$.

Proof. From $\partial_t \theta_\eta(T) = 0$ and $(\text{ curl } H_\alpha \times \nu - \text{ curl } H \times \nu)|_{t=0} = 0$ the term $\int_0^T \int_\Gamma \partial_t \theta_\eta \cdot \partial_t(\text{ curl } H_\alpha \times \nu - \text{ curl } H \times \nu)$ has to be interpreted as follows

$$\int_0^T \int_\Gamma \partial_t \theta_\eta \cdot \partial_t(\text{ curl } H_\alpha \times \nu - \text{ curl } H \times \nu) = - \int_0^T \int_\Gamma \partial_t^2 \theta_\eta \cdot (\text{ curl } H_\alpha \times \nu - \text{ curl } H \times \nu). \tag{29}$$

Next, introduce

$$\tilde{H}_{\alpha,\eta}(x,t) = H(x,t) + \int_0^t e^{-i\sqrt{\varepsilon_0}|\eta| s} v_\eta(x, t-s) \, ds, x \in \Omega, t \in (0,T). \tag{30}$$

We have

$$\begin{aligned}
&\int_0^T \int_\Gamma \Big[\theta_\eta \cdot (\text{ curl } H_\alpha \times \nu - \text{ curl } H \times \nu) + \partial_t \theta_\eta \cdot \partial_t(\text{ curl } H_\alpha \times \nu - \text{ curl } H \times \nu)\Big] = \\
&\int_0^T \int_\Gamma \Big[\theta_\eta \cdot (\text{ curl } H_\alpha \times \nu - \text{ curl } \tilde{H}_{\alpha,\eta} \times \nu) + \partial_t \theta_\eta \cdot \partial_t(\text{ curl } H_\alpha \times \nu - \text{ curl } \tilde{H}_{\alpha,\eta} \times \nu)\Big] \\
&+ \int_0^T \int_\Gamma \Big[\theta_\eta \cdot \int_0^t e^{-i\sqrt{\varepsilon_0}|\eta| s} v_\eta(x,t-s) \times \nu \, ds + \partial_t \theta_\eta \cdot \partial_t \int_0^t e^{-i\sqrt{\varepsilon_0}|\eta| s} v_\eta(x,t-s) \times \nu \, ds\Big].
\end{aligned}$$

Since θ_η satisfies the Volterra equation (25) and

$$\begin{aligned}
&\partial_t(\int_0^t e^{-i\sqrt{\varepsilon_0}|\eta| s} v_\eta(x,t-s) \times \nu \, ds) = \partial_t(e^{-i\sqrt{\varepsilon_0}|\eta| t} \int_0^t e^{i\sqrt{\varepsilon_0}|\eta| s} v_\eta(x,s) \times \nu \, ds) \\
&= i\sqrt{\varepsilon_0}|\eta| e^{-i\sqrt{\varepsilon_0}|\eta| t} \int_0^t e^{i\sqrt{\varepsilon_0}|\eta| s} v_\eta(x,s) \times \nu \, ds + v_\eta(x,t) \times \nu,
\end{aligned}$$

we obtain by integrating by parts over $(0,T)$ that

$$\begin{aligned}
&\int_0^T \int_\Gamma \Big[\theta_\eta \cdot \int_0^t e^{-i\sqrt{\varepsilon_0}|\eta| s} v_\eta(x,t-s) \times \nu \, ds + \partial_t \theta_\eta \cdot \partial_t \int_0^t e^{-i\sqrt{\varepsilon_0}|\eta| s} v_\eta(x,t-s) \times \nu \, ds\Big] \\
&= \int_0^T \int_\Gamma (v_\eta(x,t) \times \nu) \cdot (\partial_t \theta_\eta + \int_t^T \theta_\eta(s) e^{i\sqrt{\varepsilon_0}|\eta|(t-s)} \, ds) \\
&\quad -i\sqrt{\varepsilon_0}|\eta| (e^{-i\sqrt{\varepsilon_0}|\eta| t} \partial_t \theta_\eta(t)) \cdot \int_0^t e^{i\sqrt{\varepsilon_0}|\eta| s} v_\eta(x,s) \times \nu \, ds \, dt \\
&= \int_0^T \int_\Gamma v_\eta(x,t) \times \nu \cdot (\partial_t \theta_\eta + \int_t^T (\theta_\eta(s) - i\sqrt{\varepsilon_0}|\eta| \partial_t \theta_\eta(s)) e^{i\sqrt{\varepsilon_0}|\eta|(t-s)} \, ds) \, dt \\
&= \int_0^T \int_\Gamma g_\eta(x,t) \cdot (\text{ curl } v_\eta(x,t) \times \nu) \, dt
\end{aligned}$$

and so, from Proposition 3.1 we obtain

$$\int_0^T \int_\Gamma \Big[\theta_\eta \cdot (\text{ curl } H_\alpha \times \nu - \text{ curl } H \times \nu) + \partial_t \theta_\eta \cdot \partial_t (\text{ curl } H_\alpha \times \nu - \text{ curl } H \times \nu)\Big] =$$
$$\alpha^d \sum_{j=1}^m (1 - \frac{\varepsilon_j}{\varepsilon_0}) e^{2i\eta\cdot z_j} \big(\eta \times (\int_{\partial B_j} (\nu_j + (\frac{\varepsilon_j}{\varepsilon_0} - 1) \frac{\partial \Phi_j}{\partial \nu_j}|_+(y))\eta \cdot y)\big) \cdot \eta^\perp \, ds_j(y)$$
$$+ \int_0^T \int_\Gamma \Big[\theta_\eta \cdot (\text{ curl } H_\alpha \times \nu - \text{ curl } \tilde{H}_{\alpha,\eta} \times \nu) + \partial_t \theta_\eta \cdot \partial_t (\text{ curl } H_\alpha \times \nu$$
$$- \text{ curl } \tilde{H}_{\alpha,\eta} \times \nu)\Big] + o(\alpha^d).$$

In order to prove Theorem 3.1 it suffices then to show that

$$\int_0^T \int_\Gamma \Big[\theta_\eta \cdot (\text{ curl } H_\alpha \times \nu - \text{ curl } \tilde{H}_{\alpha,\eta} \times \nu) + \partial_t \theta_\eta \cdot \partial_t (\text{ curl } H_\alpha \times \nu - \text{ curl } \tilde{H}_{\alpha,\eta} \times \nu)\Big] = o(\alpha^d). \tag{31}$$

Since

$$\begin{cases} (\partial_t^2 - \text{ curl } \frac{1}{\varepsilon_0} \text{ curl })(\int_0^t e^{-i\sqrt{\varepsilon_0}|\eta| s} v_\eta(x, t-s) \, ds) \\ = \sum_{j=1}^m i(1 - \frac{\varepsilon_j}{\varepsilon_0})\eta \times (\nu_j + (\frac{\varepsilon_j}{\varepsilon_0} - 1) \frac{\partial \Phi_j}{\partial \nu_j}|_+(y)) e^{i\eta\cdot z_j} \delta_{\partial(z_j + \alpha B_j)} e^{-i\sqrt{\varepsilon_0}|\eta| t} \quad \text{in } \Omega \times (0,T), \\ (\int_0^t e^{-i\sqrt{\varepsilon_0}|\eta| s} v_\eta(x, t-s) \, ds)|_{t=0} = 0, \partial_t (\int_0^t e^{-i\sqrt{\varepsilon_0}|\eta| s} v_\eta(x, t-s) \, ds)|_{t=0} = 0 \quad \text{in } \Omega, \\ (\int_0^t e^{-i\sqrt{\varepsilon_0}|\eta| s} v_\eta(x, t-s) \, ds) \times \nu|_{\partial\Omega \times (0,T)} = 0, \end{cases}$$

it follows from Theorem 2.1 that

$$\begin{cases} (\partial_t^2 - \text{ curl } \frac{1}{\varepsilon_0} \text{ curl })(H_\alpha - \tilde{H}_{\alpha,\eta}) = O(\alpha^d) \in Y(\Omega), \quad \text{in } \Omega \times (0,T), \\ (H_\alpha - \tilde{H}_{\alpha,\eta})|_{t=0} = 0, \partial_t (H_\alpha - \tilde{H}_{\alpha,\eta})|_{t=0} = 0 \quad \text{in } \Omega, \\ (H_\alpha - \tilde{H}_{\alpha,\eta}) \times \nu|_{\partial\Omega \times (0,T)} = 0. \end{cases}$$

Following the proof of Proposition 2.1, we immediately obtain

$$||H_\alpha - \tilde{H}_{\alpha,\eta}||_{\mathbf{L}^2(\Omega)} = O(\alpha^d), t \in (0,T), x \in \Omega,$$

where $O(\alpha^d)$, $d = 2, 3$ is independent of the points $\{z_j\}_{j=1}^m$. To prove (31) it suffices then from (29) to show that the following estimate holds

$$|| \text{ curl } H_\alpha \times \nu - \text{ curl } \tilde{H}_{\alpha,\eta} \times \nu ||_{L^2(0,T; TL^2(\Gamma))} = O(\alpha^d).$$

Let θ be given in $\mathcal{C}_0^\infty(]0,T[)$ and define

$$\hat{\tilde{H}}_{\alpha,\eta}(x) = \int_0^T \tilde{H}_{\alpha,\eta}(x,t)\theta(t) \, dt$$

and

$$\hat{H}_\alpha(x) = \int_0^T H_\alpha(x,t)\theta(t)\,dt.$$

From definition (30) we can write

$$\begin{cases} (\hat{H}_\alpha - \hat{\hat{H}}_\alpha) \in \mathbf{H}^1(\Omega), \\ \text{ curl } \text{ curl } (\hat{H}_\alpha - \hat{\hat{H}}_\alpha) = O(\alpha^d) \in Y(\Omega) \quad \text{in } \Omega, \\ \text{div } (\hat{H}_\alpha - \hat{\hat{H}}_\alpha) = 0 \quad \text{in } \Omega, \\ (\hat{H}_\alpha - \hat{\hat{H}}_\alpha) \times \nu|_{\partial\Omega} = 0. \end{cases} \tag{32}$$

In the spirit of the standard elliptic regularity [7] we deduce for the boundary value problem (32) that

$$||\text{ curl } (\hat{H}_\alpha - \hat{\hat{H}}_\alpha) \times \nu||_{\mathbf{L}^2(\Gamma)} = O(\alpha^d),$$

for all $\theta \in \mathcal{C}_0^\infty(]0,T[)$; whence

$$||\text{ curl } (H_\alpha - \hat{H}_\alpha) \times \nu||_{\mathbf{L}^2(\Gamma)} = O(\alpha^d) \text{ a.e. in } t \in (0,T),$$

and so, the desired estimate (27) holds. The proof of Theorem 3.1 is then over. □

Our identification procedure is deeply based on Theorem 3.1. Let us neglect the asymptotically small remainder in the asymptotic formula (27), and define $\aleph_\alpha(\eta)$ by

$$\aleph_\alpha(\eta) = \int_0^T \int_\Gamma \Big[\theta_\eta \cdot (\text{ curl } (H_\alpha - H) \times \nu) + \partial_t\theta_\eta \cdot \partial_t(\text{ curl } (H_\alpha - H) \times \nu)\Big].$$

Recall that the function $e^{2i\eta\cdot z_j}$ is exactly the Fourier Transform (up to a multiplicative constant) of the Dirac function δ_{-2z_j} (a point mass located at $-2z_j$). From Theorem 3.1 it follows that the function $e^{2i\eta\cdot z_j}$ is (approximately) the Fourier Transform of a linear combination of derivatives of point masses, or

$$\breve{\aleph}_\alpha(\eta) \approx \alpha^d \sum_{j=1}^m L_j \delta_{-2z_j},$$

where L_j is a second order constant coefficient, differential operator whose coefficients depend on the polarization tensor M_j defined by (28) (see [6] for its properties) and $\breve{\aleph}_\alpha(\eta)$ represents the inverse Fourier Transform of $\aleph_\alpha(\eta)$.

The method of reconstruction consists in sampling values of $\breve{\aleph}_\alpha(\eta)$ at some discrete set of points and then calculating the corresponding discrete inverse Fourier Transform. After a rescaling the support of this discrete inverse Fourier Transform yields the location of the small inhomogeneities $\mathcal{B}_\alpha$. Once the locations are known we may calculate the polarization tensors $(M_j)_{j=1}^m$ by solving an appropriate linear system arising from (27). This procedure generalizes the approach developed in [3] for the two-dimensional (time-independent) inverse conductivity problem and generalize the results in [1] to the full time-dependent Maxwell's equations.

References

[1] H. Ammari, *An inverse initial boundary value problem for the wave equation in the presence of imperfections of small volume.* SIAM J. Control Optim. 41 (2003), 1194-1211.

[2] H. Ammari, H. Kang, E. Kim, and M. Lim, *Reconstruction of closely spaced small inclusions.*SIAM J. Numer. Anal. 42, no. 6, (2005), 2408-2428.

[3] H. Ammari, S. Moskow, and M. Vogelius, *Boundary integral formulas for the reconstruction of electromagnetic imperfections of small diameter,* ESAIM: COCV 9 (2003), 49-66.

[4] H. Ammari, M. Vogelius, and D. Volkov, *Asymptotic formulas for perturbations in the electromagnetic fields due to the presence of inhomogeneities of small diameter II.The full Maxwell equations*, J. Math. Pures Appl. 80 (2001), 769-814.

[5] C. Bardos, G. Lebeau, and J. Rauch, *Sharp sufficient conditions for the observation, control and stabilization of waves from the boundary,* SIAM J. Control Opt. 30 (1992), 1024-1065.

[6] D. J. Cedio-Fengya, S. Moskow, and M. Vogelius, *Identification of conductivity imperfections of small diameter by boundary measurements. Continuous dependence and computional reconstruction,* Inverse Problems 14 (1998), 553-595.

[7] L. C. Evans, *Partial Differential Equations,* Graduate Studies in Mathematics, AMS, 1998, Providence, Rhode Island.

[8] A. Friedman and M. Vogelius, *Identification of small inhomogeneities of extreme conductivity by boundary measurements: a theorem on continuous dependence*, Arch. Rat. Mech. Anal. 105 (1989), 299-326.

[9] K. A. Kime, *Boundary controllability of Maxwell's equations in a spherical region,* SIAM J. Control Optim. 28 (1990), 294-319.

[10] V. Komornik, *Boundary stabilization, observation and control of Maxwell's equations,* Panamer. Math. J. 4 (1994), 47-61.

[11] J. E. Lagnese, *Exact boundary controllability of Maxwell's equations in a general region,* SIAM J. Control Optim. 27 (1989), 374-388.

[12] J. L. Lions, *Contrôlabilité exacte, Perturbations et Stabilisation de Systèmes Distribués, Tome 1, Contrôlabilité Exacte,* Masson 1988, Paris.

[13] J. L. Lions and E. Magenes, *Nonhomogeneous Boundary Value Problems and Applications,* Vol. 1, Springer, 1972.

[14] S. R. McDowall, *An electromagnetic inverse problem in chiral media,* Trans. Amer. Math. Soc. 352 (2000), 2993-3013.

[15] S. Nicaise, *Exact boundary controllability of Maxwell's equations in heteregeneous media and an application to an inverse source problem,* SIAM J. Control Optim. 38 (2000), 1145-1170.

[16] V. G. Romanov and S. I. Kabanikhin, *Inverse Problems for Maxwell's Equations,* Inverse and Ill-posed Problems Series, VSP, Utrecht, 1994.

[17] J. Sylvester and G. Uhlmann, *A global uniqueness theorem for an inverse boundary value problem,* Ann. Math. 125 (1987), 153-169.

[18] E. Somersalo, D. Isaacson, and M. Cheney, *A linearized inverse boundary value problem for Maxwell's equations,* J. Comput. Appl. Math. 42 (1992), 123-136.

[19] M. Vogelius and D. Volkov, *Asymptotic formulas for perturbations in the electromagnetic fields due to the presence of inhomogeneities*, Math. Model. Numer. Anal. 34 (2000), 723-748.

[20] D. Volkov, *An inverse problem for the time harmonic Maxwell equations,* PhD thesis, Rutgers University, New Brunswick, NJ,2001.